Last time:

① $\left(\frac{3}{7}, \frac{6}{21}, \frac{9}{63}, \frac{12}{189}, \ldots\right)$ — A_1, A_2, A_3

② $\left(\frac{3}{2}, \frac{8}{3}, \frac{15}{4}, \frac{24}{5}, \frac{35}{6}, \ldots\right)$ — B_1, B_2, B_3, B_4

③ $(20, 1, 21, 22, 43, 65, 108, \ldots)$ — C_1, C_2, C_3, C_4, C_5, C_6

1A for $k \geq 1$, $A_k = \frac{3k}{7 \cdot 3^{k-1}}$

1B $A_1 = \frac{3}{7}$, for $k \geq 2$, $\cancel{A_k = A_{n-1} - \frac{3}{21} = A_{n-1} - \frac{1}{7}}$

~~Could also write~~ $\cancel{A_{n+1} = A_n - \frac{1}{7}}$

2A $\boxed{B_n = \frac{n(n+2)}{(n+1)} \quad n \geq 1}$ closed form.

$B_1 = \frac{1 \cdot 3}{2}$

$B_2 = \frac{2 \cdot 4}{3}$

$B_3 = \frac{3 \cdot 5}{4}$ ✓

2B

3B
$$\left.\begin{aligned} C_n &= C_{n-2} + C_{n-1} \\ C_1 &= 20 \\ C_2 &= 1 \end{aligned}\right] \text{ recursive}$$

My solutions:

① Closed form: For $k \geq 1$, $A_k = \frac{3k}{7 \cdot 3^{k-1}}$

$= \frac{k}{7 \cdot 3^{k-2}}$

Recursive form: $\begin{cases} A_1 = \frac{3}{7} \\ A_{n+1} = \frac{A_n}{3}\left(\frac{n+1}{n}\right) = \frac{A_n}{3} + \frac{A_n}{3n} \end{cases}$

② Closed form: $B_j = \frac{(j+1)^2 - 1}{j+1} = \frac{j^2 + 2j}{j+1} = \frac{j(j+2)}{j+1}$

$B_j = \frac{j(j+2)}{j+1} = \frac{1}{(j-1)(j+1)/j} \cdot (j+2)(j-1) = \frac{(j+2)(j-1)}{B_{j-1}}$

Recursive: $B_1 = \frac{3}{2}$, $B_j = \frac{(j+2)(j-1)}{B_{j-1}}$ for $j \geq 2$.

③ Recursive: $C_1 = 20$, $C_2 = 1$, $C_{j+1} = C_j + C_{j-1}$ for $j \geq 1$.

$$\begin{pmatrix} C_{n+1} \\ C_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_n \\ C_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 0 \\ 0 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \end{pmatrix}^{-1} \cdot \begin{pmatrix} C_n \\ C_{n-1} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} C_{n+1} \\ C_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} C_2 \\ C_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 20 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 20 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \end{pmatrix} \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ +\frac{1+\sqrt{5}}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 20 \end{pmatrix}$$

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \end{pmatrix} \begin{pmatrix} \frac{-41+\sqrt{5}}{2} \\ \frac{41+\sqrt{5}}{2} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{-41+\sqrt{5}}{2}\right) \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{41+\sqrt{5}}{2}\right) \end{pmatrix}$$

$$\Rightarrow \boxed{C_n = \frac{-1}{\sqrt{5}} \frac{(1+\sqrt{5})}{2} \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \left(\frac{-41+\sqrt{5}}{2}\right) + \frac{1}{\sqrt{5}} \left(\frac{-1+\sqrt{5}}{2}\right) \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \left(\frac{41+\sqrt{5}}{2}\right) \quad \text{for } n \geq 1 \quad \text{closed form.}}$$

More examples of sequence and their limits.

Useful Defns and facts:

A sequence $(x_n)_{n\geq 1}$ is **bounded** if there exists a number $R > 0$ such that $|x_n| < R$ for all $n \geq 1$.

ie $-R < x_n < R$ for all n.

A sequence $(y_n)_{n\geq 1}$ is <u>monotone</u> if $y_n \geq y_{n-1}$ for all $n \geq 2$ [monotone increasing]

or if $y_n \leq y_{n-1}$ for all $n \geq 2$ [monotone decreasing].

<u>Thm</u> (Monotone Convergence Theorem) MCT

Every bounded monotone sequence converges.

(A sequence $(x_n)_{n\geq 1}$ converges if $\lim_{n\to\infty} x_n$ exists and is a finite # L.)

Examples:

$$(2, 4, 18, 29, 35, \ldots)$$

This sequence looks like it is unbounded & monotone increasing & looks like it diverges (does not converge).

$$\left(\overset{x_1}{1.1}, \overset{x_2}{1.11}, \overset{x_3}{1.111}, 1.1111, \ldots\right)$$

This sequence is bounded, because

$$-2 < 1 < x_n < 2 \quad \text{for all } n.$$

It is also monotone increasing.

By the MCT, it converges.

$$\lim_{n\to\infty} x_n = \underset{1.1111111\ldots}{1.1\overline{1}} = \frac{10}{9}.$$

Let $y_n = (-1)^n$.

This sequence is bounded:

$-2 < -1 \leq y_n \leq 1 < 2$ for all n.

$(-1, 1, -1, 1, \ldots)$

The sequence is not monotone.

It does not converge — it does not get close to any particular #.

$$z_n = \frac{(-1)^n}{n^2}$$

$(-1, \frac{1}{4}, \frac{-1}{9}, \ldots)$

The sequence is bdd

$-2 < -1 \leq z_n < 1 < 2$ for all n.

Not monotone.

$$\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$$

It converges.

$$\frac{-1}{n^2} \le \frac{(-1)^n}{n^2} \le \frac{1}{n^2}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$

(converse to MCT is false. Sequences can converge even if they are not monotone.)

$\boxed{\text{EX}}$ Find $1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{\ddots}}}}$

This is an example of a "continued fraction." ← it means the limit of a sequence.

$x_1 = 1$

$x_2 = 1+\frac{2}{1} = 3$

$x_3 = 1+\cfrac{2}{1+\frac{2}{1}} = 5/3$

$x_4 = 1+\cfrac{2}{1+\cfrac{2}{1+\frac{2}{1}}} = \frac{11}{5}$ ..

$\left(1, 3, \frac{5}{3}, \frac{11}{5}, \dots\right)$

What is $\lim_{k \to \infty} x_k$? That is the value of the continued fraction.

Recursive formula

$$x_1 = 1$$

$$x_{n+1} = 1 + \frac{2}{x_n} \quad \text{for } n \geq 1$$

If this sequence has a limit, then

$$\lim x_{n+1} = \lim \left(1 + \frac{2}{x_n}\right)$$

$$\Rightarrow \lim x_{n+1} = 1 + \frac{2}{\lim x_n}.$$

Let $L = \lim x_n = \lim x_{n+1}$

$$\Rightarrow L = 1 + \frac{2}{L}$$

Solving for L:

$$L^2 = L + 2$$

$$L^2 - L - 2 = 0$$

$$(L-2)(L+1) = 0$$

$$\Rightarrow L = 2 \text{ or } L = -1.$$

Note: If $x_1 = 1$, then $x_n \geq 1$ for all n.

Proof: Assume this has been shown for some x_n.

$$x_{n+1} = 1 + \frac{2}{x_n} \geq 1$$

So $x_{n+1} \geq 1$ also. $\blacksquare$

So since $x_n \geq 1$ for all n,

$L = \lim x_n = 2$ or -1

$\Rightarrow L = 2$ is the only possible limit.

Let's check this out.

Type some Sage code below and press Evaluate.

```
f(x)=1+2/x
s=[1]
for j in range(20): #means j= 0,1,2,...19;
    la=s[-1] # means the last number in the list
    nxt=f(la).n() # y.n() makes y numerical
    s+=[nxt]  # adds nxt to the list s

print(s)
```

Evaluate

```
73260, 1.99963383376053, 2.00018311664530, 1.99990845005951, 2.000
```

About

This shows that our sequence does converge to 2!

What happens if we use the same recursive formula but have different x_i's?